Characterization of Circular-Arc Graphs

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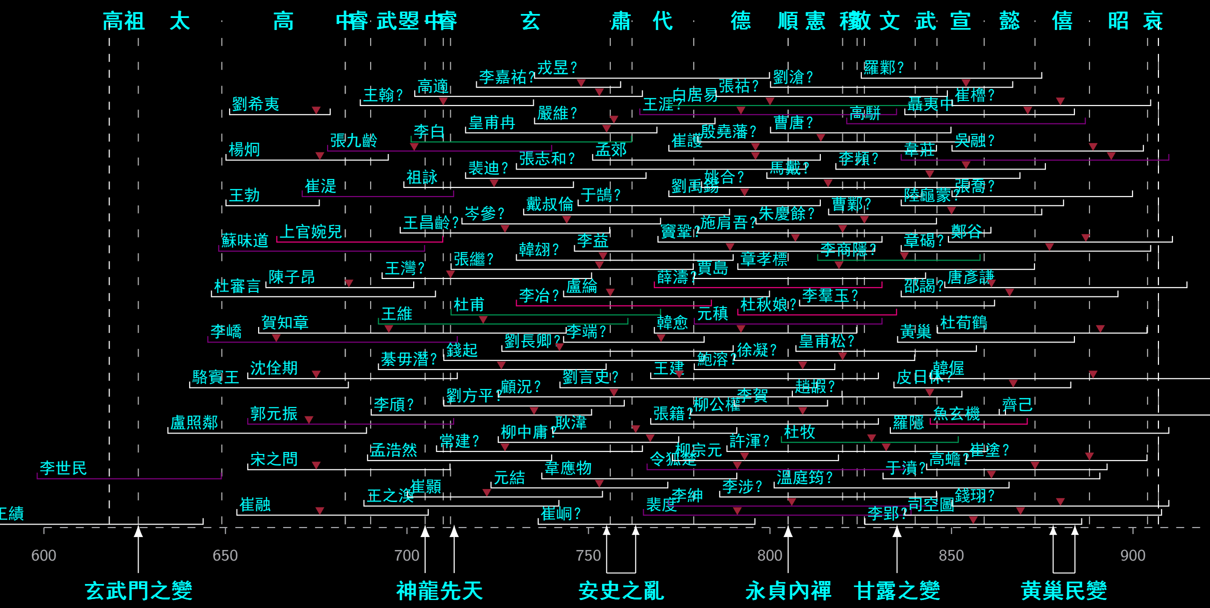
上海交通大学 2025年5月20日



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杜秋娘？
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黃巢
李嶠
李端？
皇甫松？
劉長卿？
錢起
徐凝？
綦毋潛？
鮑溶？
沈佺期
王建
韓偓
駱賓王
劉言史？
皮日休？
顧況？
趙嘏？
劉方平？
李賀
李頎？
柳公權
齊己
郭元振
張籍？
魚玄機
盧照鄰
耿湋
羅隱
柳中庸？
杜牧
常建？
許渾？
孟浩然
柳宗元
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宋之問
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李世民
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于濆？
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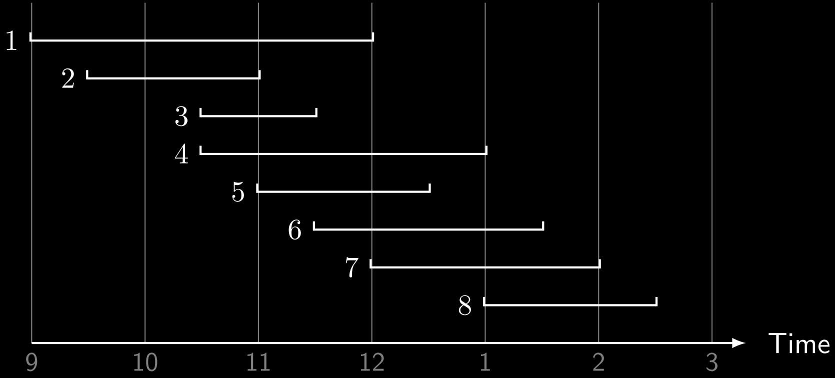
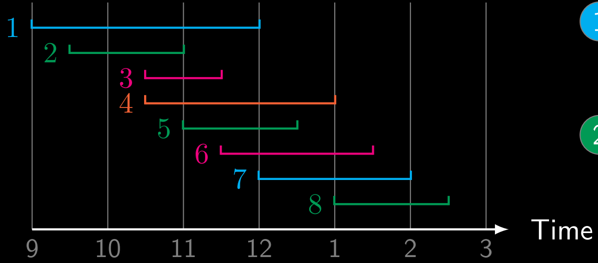
玄武門之變
神龍先天
安史之亂
永貞內禪
甘露之變
黃巢民變

How many meeting rooms do we need to accommodate the following meeting plan?

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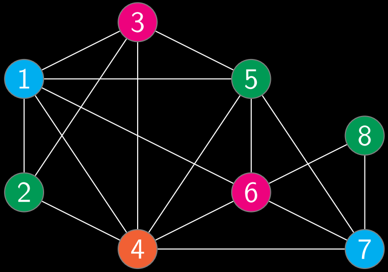
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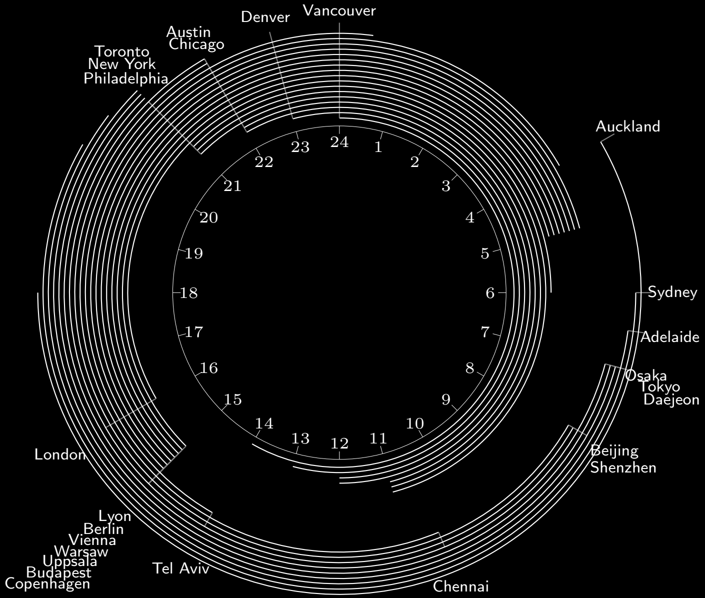

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Tel Aviv
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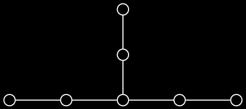
Theorem (Feder, Hell, and Huang 1998, 1999).
The list homomorphism problem (whether an input graph G admits a list homomorphism to a fixed graph H) can be solved in polynomial time if and only if H is an **interval graph** (for reflexive graphs) or a special kind of **circular-arc graph** (when no loops are allowed).

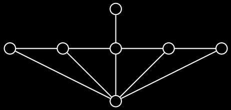
Chudnovsky and Seymour [2005–2012]: "every connected claw-free graph can be obtained from one of the basic claw-free graphs by simple expansion operations," one of which is a special type of **circular-arc graphs**.

The Problem

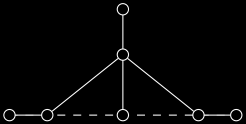
Theorem (Lekkerkerker and Boland 1962). A graph is an interval graph if and only if it doesn't contain any hole $(C_4, C_5, \ldots)$ or any graph below as an induced subgraph.



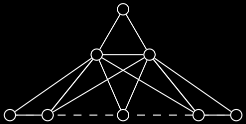
(a) long claw



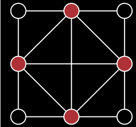
(b) whipping top



(c) †



(d) ‡



A dashed lines indicate a path of any length.

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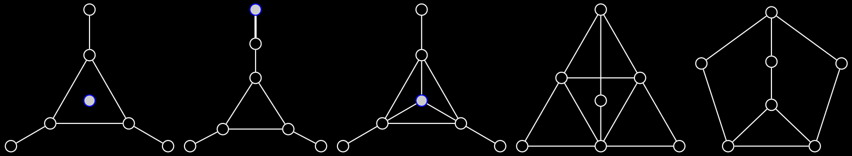
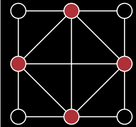
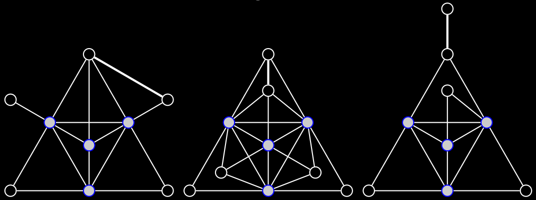
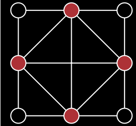




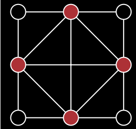
The following are not circular-arc graphs. Which of them are minimal?





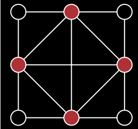
A graph is *chordal* if all induced cycles are triangles.

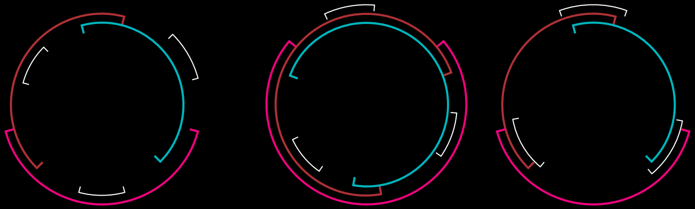
- chordal: an intersection graph of subtrees of a tree
- interval = chordal ∩ normal Helly circular-arc ⊂ chordal ∩ circular-arc

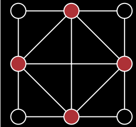


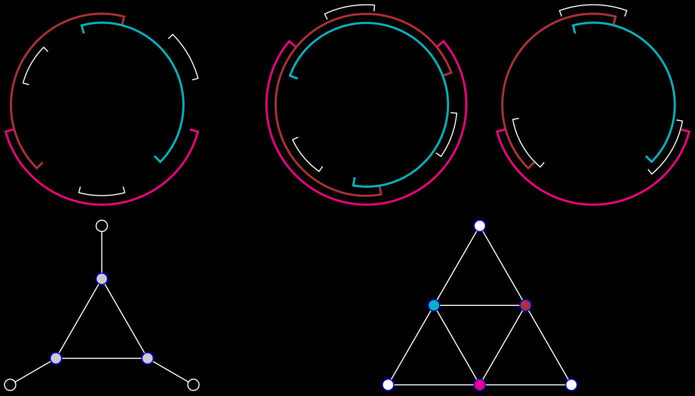
Durán, Grippo, and Safe, *Structural results on circular-arc graphs and circle graphs: A survey and the main open problems*

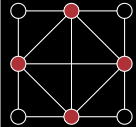
Problem 1. Give a forbidden induced subgraph characterization for circular-arc graphs within the class of chordal graphs. This would extend the characterizations in [Bonomo et al. 2009, Francis et al. 2014] of circular-arc graphs within claw-free chordal graphs and $5K_1$-free chordal graphs, respectively.





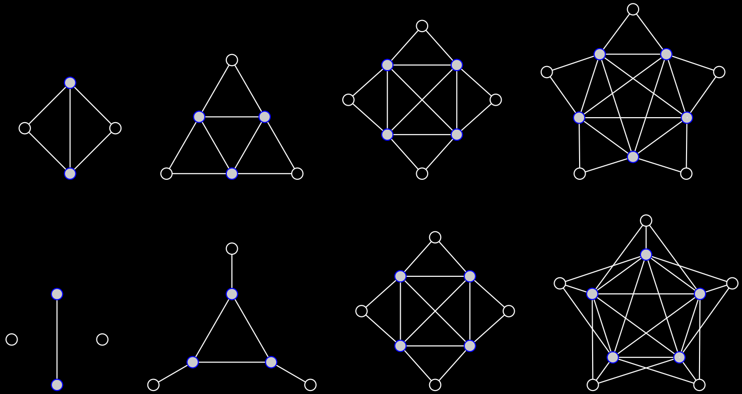


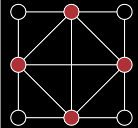




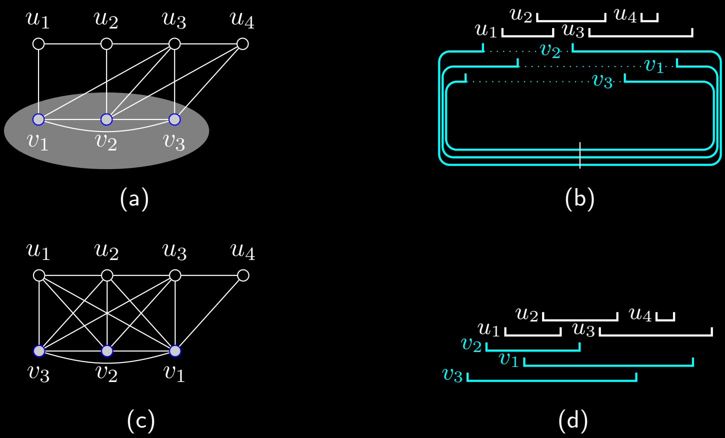
← Does this graph has a unique model (like #1) or multiple ones (like #2)?

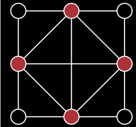
k-suns and their complements ($k = 2, 3, 4, 5$)





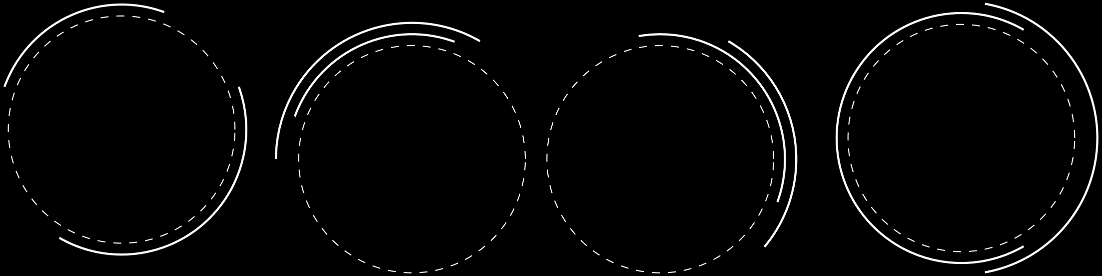
McConnel Flipping



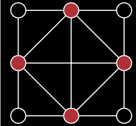


trivial: a $\to$ b $\to$ d $\to$ c

magical: a $\to$ c $\to$ d $\to$ b



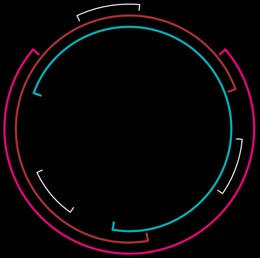
(a) independent (b) containment (c) cross (d) double overlap

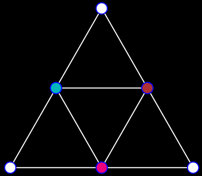


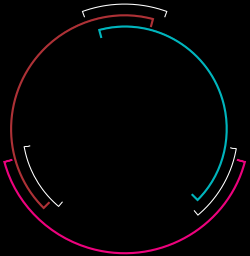
Cross-rigid models

A circular-arc model is *cross-rigid* if for every pair v_1 and v_2.

- The arc $A(v_1)$ contains the arc $A(v_2)$ whenever $N[v_2] \subseteq N[v_1]$.
- The arcs $A(v_1)$ and $A(v_2)$ double overlap whenever possible.

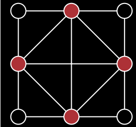




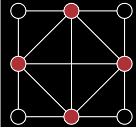


Which is cross-rigid?

What is a cross-rigid model for P_4?



Theorem (Hsu 95).
Any circular-arc graph with no universal vertex admits a cross-rigid model.

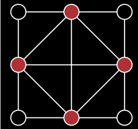


Theorem (McConnel 2001). Circular-arc graphs can be recognized in linear time.

1. find a "suitable" clique K;
2. "flip" K;
3. build a special interval model;
4. retrieve a circular-arc model.

$$[\mathrm{lp}, \mathrm{rp}] \longrightarrow [\mathrm{rp}, \mathrm{lp}]$$

A crucial trick: we can flip the arcs without the arcs.

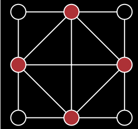


The auxiliary graph G^K

The vertex set is $V(G)$, and a pair of vertices u, v are adjacent in G^K if

- $u, v \in V(G) \setminus K$ — $uv \in E(G)$
- $u, v \in K$ — $N_G(u) \cup N_G(v) \neq V(G)$
- $u \in K$ and $v \in V(G) \setminus K$ — $N_G[v] \not\subseteq N_G[u]$

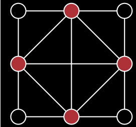
they are not adjacent, or there exists a vertex adjacent to v but not u in G.



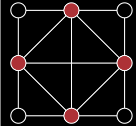
Theorem (C., Krawczyk, 2024$^+$).
A chordal graph G is a circular-arc graph if and only if there exists a clique K such that the graph G^K admits an interval model in which for all $x \in K$ and $y \notin K$,

$$I(y) \subseteq I(x) \iff xy \notin E(G).$$

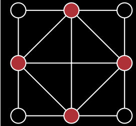
Proof. $\begin{cases} [\mathrm{rp}(x), \mathrm{lp}(x)] & \text{if } x \in K, \\ [\mathrm{lp}(x), \mathrm{rp}(x)] & \text{if } x \notin K. \end{cases}$ □



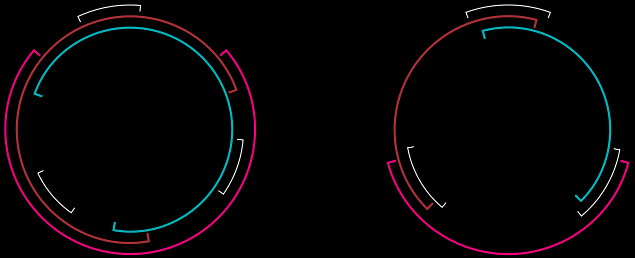
Split ∩ Helly Circular-arc

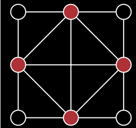


Spinrad: “split graphs …
often seem to be at the core of algorithms and proofs of difficulty for chordal graphs.”



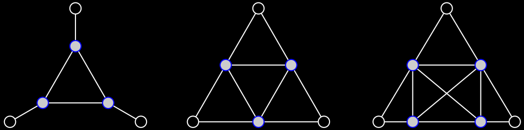
A circular-arc model is *Helly* if the arcs for every maximal clique have a shared point.





Theorem (Joeris et al. 2011).

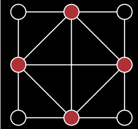
All cross-rigid models of Helly circular-arc graphs are Helly.

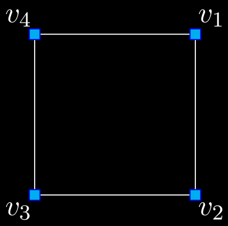


(a) net (b) sun (c) rising sun

Theorem (C., Krawczyk, 2024$^+$).
Let G be a split graph with a split partition $K \uplus S$. G is a Helly circular-arc graph if and only if G^K is an interval graph.

Proof. Two vertices $v \in K$ and $u \in V(G) \setminus K$ are adjacent if and only if they are not adjacent in G. □

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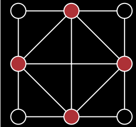

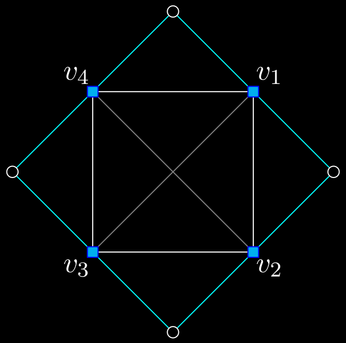
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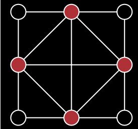
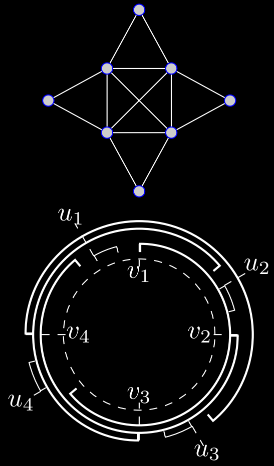
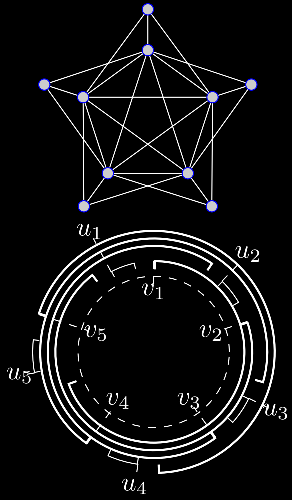
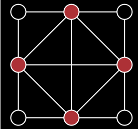
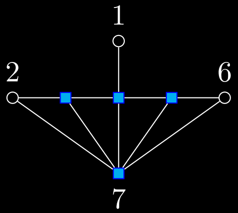








Minimal split graphs that are not Helly circular-arc graphs ($n = 4, 5, \ldots$).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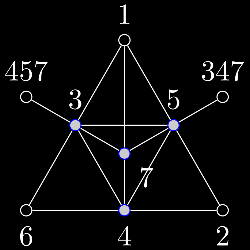
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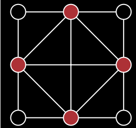
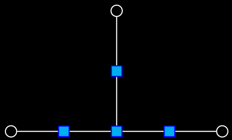
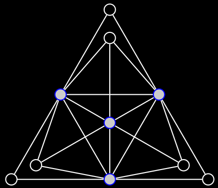
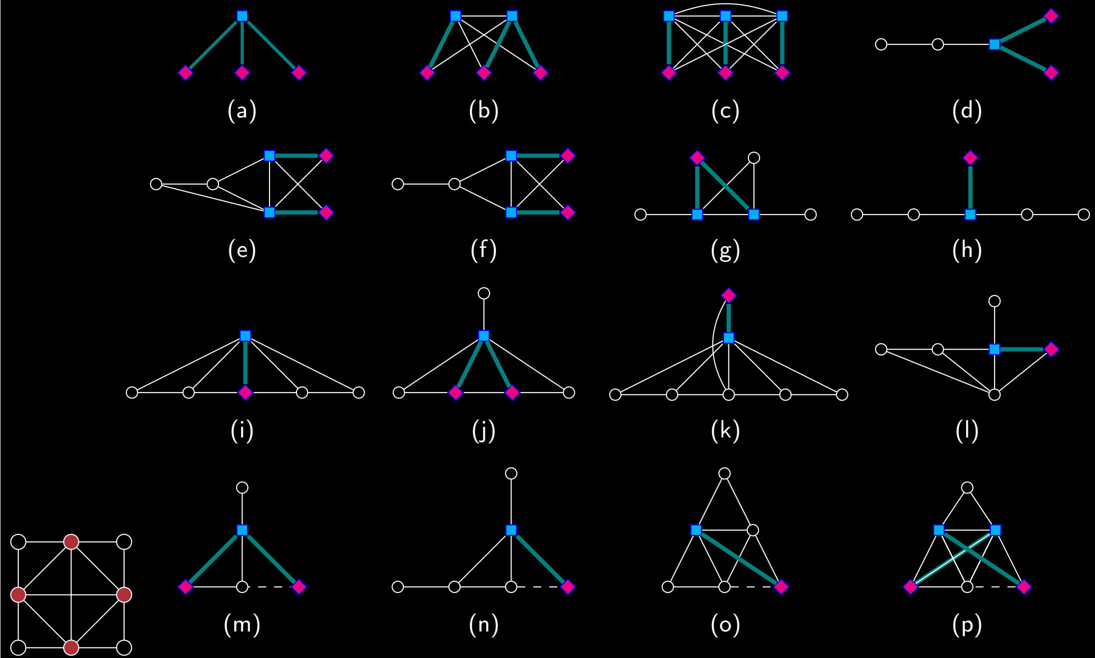






Chordal $\cap$ Circular-arc

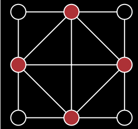

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)
(m)
(n)
(o)
(p)

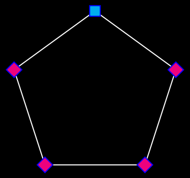
Theorem (C., Krawczyk 2024^{+}).
The following are equivalent on a chordal graph G.

i) The graph G is a circular-arc graph.

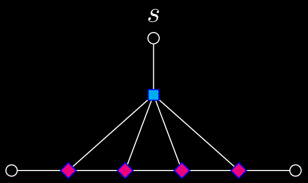
ii) For every simplicial vertex s, the graph $G^{N[s]}$ does not contain any annotated copy of forbidden configurations.

iii) There exists a simplicial vertex s such that $G^{N[s]}$ does not contain any annotated copy of forbidden configurations.



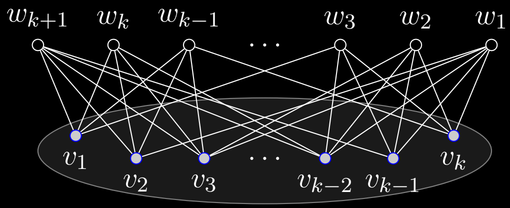


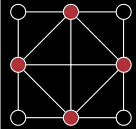
$G^{N[s]}$



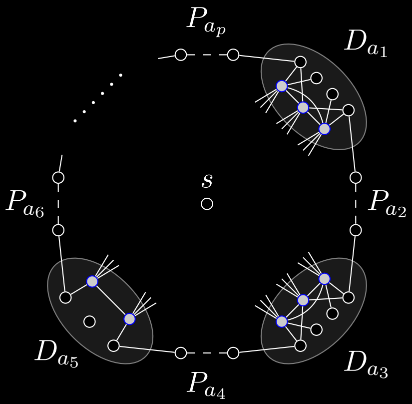
G

The gadget D_k

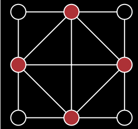


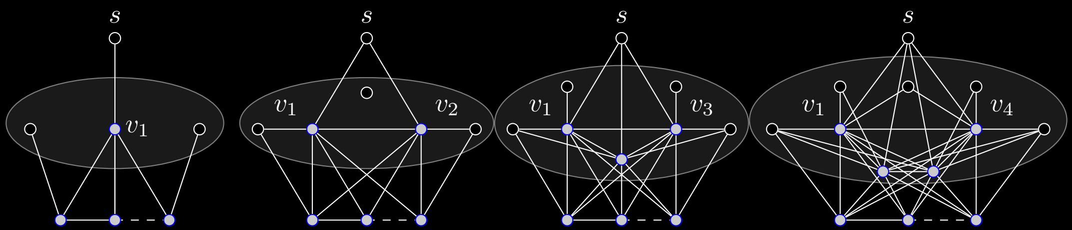


The graph $O(a_1, a_2, \ldots a_p)$

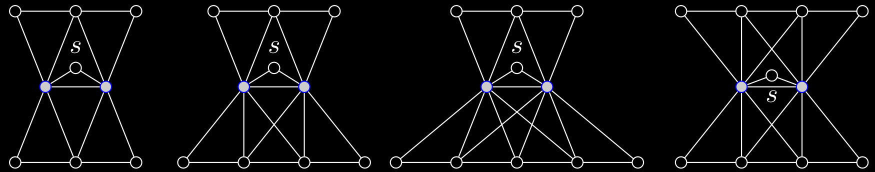


Inside each shadowed ellipse is a gadget.
Each solid vertex is adjacent to all vertices not in the same gadget.

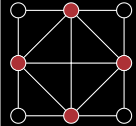


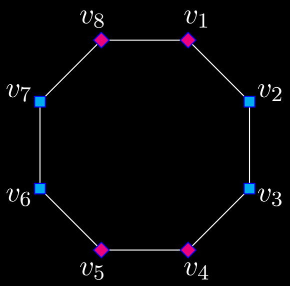


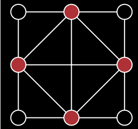
(a) $O(1,q)$ (b) $O(2,q)$ (c) $O(3,q)$ (d) $O(4,q)$

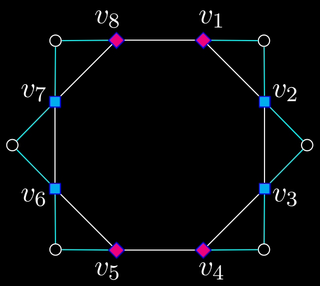


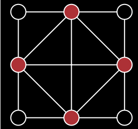
(a) $O(1,1,1,1)$ (b) $O(1,1,1,2)$ (c) $O(1,1,1,3)$ (d) $O(1,2,1,2)$



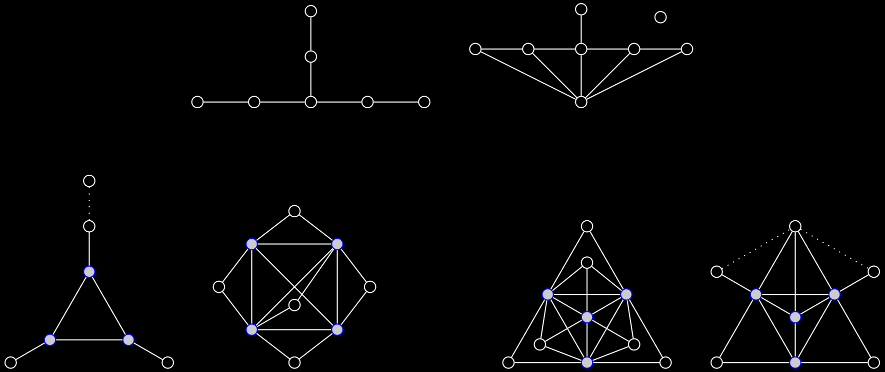

v_8
v_1
v_7
v_2
v_6
v_3
v_5
v_4

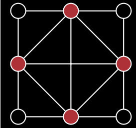



v_8
v_1
v_7
v_2
v_6
v_3
v_5
v_4



Other minimal chordal forbidden induced subgraphs

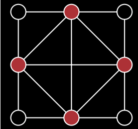




A dotted lines indicate a potential edge (which may or may not be present).

The general case:

- The forbidden configurations.
- How to find a “suitable” clique.
- Labor work.

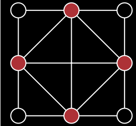


Joint work with Tomasz Krawczyk:

- Characterization of circular-arc graphs: I. split graphs arXiv:2403.01947
- Characterization of circular-arc graphs: II. McConnell flipping arXiv:2408.10892
- Characterization of circular-arc graphs: III. chordal graphs arXiv:2409.02733

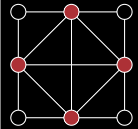
Future work

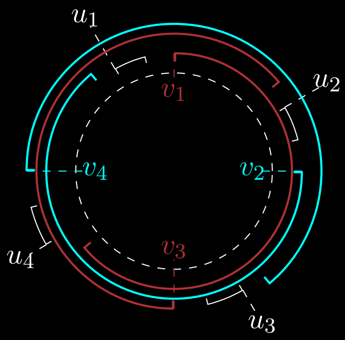
- Combine our result with the characterization of Francis, Hell and Stacho [2015].
- A certifying recognition algorithm.



Further open problems

- The complexity of counting (perfect) matchings on circular-arc graphs.
- Can circular-arc graphs be canonized in logspace?
  - Interval graphs [Köbler, Kuhnert, Laubner, Verbitsky, SIAM J. Comput. 2011]
  - Helly circular-arc graphs [Köbler, Kuhnert, Verbitsky, Inform. Comput. 2016]
  - C_4-free circular-arc graphs [Chandoo, Algorithmica 2018]
  - Bulletin of EATCS, 107:43–71, June 2012





Thanks!

I belong to a clan, or a clique, or a family, or a connection, or whatever you like to call it. *Little Dorrit*