Chordal Editing is Fixed-Parameter Tractable

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STACS 2014   Lyon, Frace

March 7, 2014

Graph modification problems

For every graph class $\mathcal{G}$ (planar, chordal, interval, etc.), we can study:

Definition ($\mathcal{G}$-graph modification problem)

Input: a graph G of size n and a nonnegative integer k
Task: find $\leq k$ modifications that transform G into a graph in $\mathcal{G}$?

Typical modification operations:

- removing edges,
- adding edges, or
- removing vertices.

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In other words, the question is if G belongs to the class

- $\mathcal{G} + ke$: a graph from $\mathcal{G}$ with k extra edges;
- $\mathcal{G} - ke$: a graph from $\mathcal{G}$ with k missing edges;
- $\mathcal{G} + kv$: a graph from $\mathcal{G}$ with k extra vertices.

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All variations are FPT recognizable when $\mathcal{G}$ has a finite set of obstructions.

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Other FPT-recognizable classes

bipartite $+ kv$ (bipartite $+ ke$) [Reed et al., 2003]; DAG $+ kv$ (DAG $+ ke$) [Chen et al., 2008]; interval $- ke$ [Heggernes et al., 2007]; interval $+ kv$ [C. & Marx, 2014]; interval $+ ke$ [C., 2014].

Modification to chordal graphs

Theorem (Cai, 1996; Kaplan et al., 1994)

Recognizing $chordal - ke$ is FPT.

Theorem (Marx, 2006)

Recognizing $chordal + kv$ and $chordal + ke$ is FPT.

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New result

Recognizing $chordal + k_1 v + k_2 e - k_3 e$ is FPT.

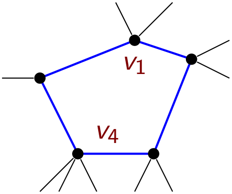
[Implication] The following problem is FPT:
at most k edge additions and deletions.
Try all combinations of (k_2, k_3) s.t. $k_2 + k_3 = k$ and $k_1 = 0$.

Chordal graphs

Definition

a graph is chordal if each of its cycles of four or more vertices has a chord.

chord: an edge connecting two non-consecutive vertices of the cycle.



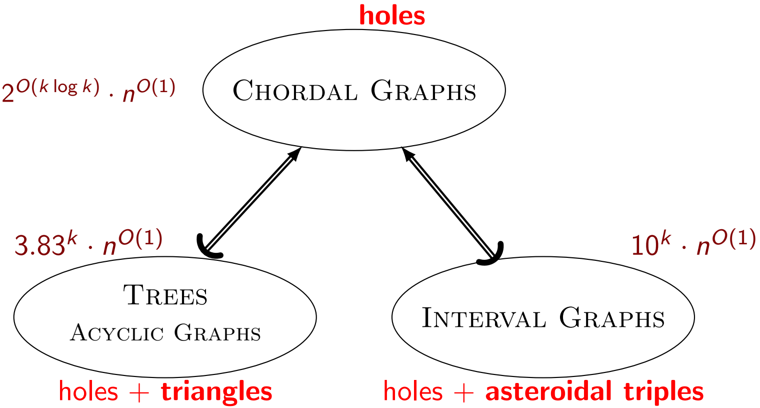
Hole: a cycle of length ≥ 4 w/o chords.

Theorem

A chordal graph is the Intersection graph of subtrees in a tree.

interval $\subset$ chordal $\subset$ perfect.

Holes and others



Techniques employed by previous results

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Completion [Cai; Kaplan et al.]

- To fill a hole H, we need at least $|H| - 3$ edges.
- We branch on adding one of $|H|(|H| - 3)/2$ chords.

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Deletion [Marx]

- If the treewidth of G is large, then we can find an **irrelevant** vertex.
- Otherwise, we can apply **Courcelle's Theorem** to the bounded-treewidth grpah.

Chordal editing set

$V_- \subseteq V(G)$ and $E_- \subseteq E(G)$ and $E_+ \subseteq V(G)^2 \setminus E(G)$

Definition

(V_-, E_-, E_+) is a *chordal editing set* of G if the deletion of V_- and E_- and the addition of E_+, applied successively, make G chordal.

- Requirement: $|V_-| \leq k_1$; $|E_-| \leq k_2$; $|E_+| \leq k_3$.
- it does not make sense to ask for $|V_-| + |E_-| + |E_+| \leq k_1 + k_2 + k_3$; otherwise, it degenerates to vertex deletion.
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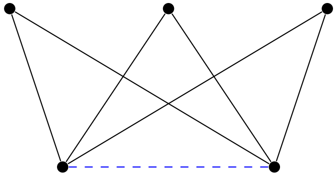

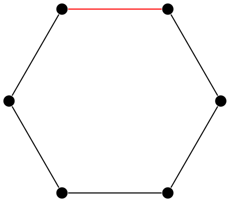

Iterative compression

Instead of solving the problem:

CHORDAL EDITING $(G, (k_1, k_2, k_3))$

Input: A graph G, nonnegative integers (k_1, k_2, k_3).

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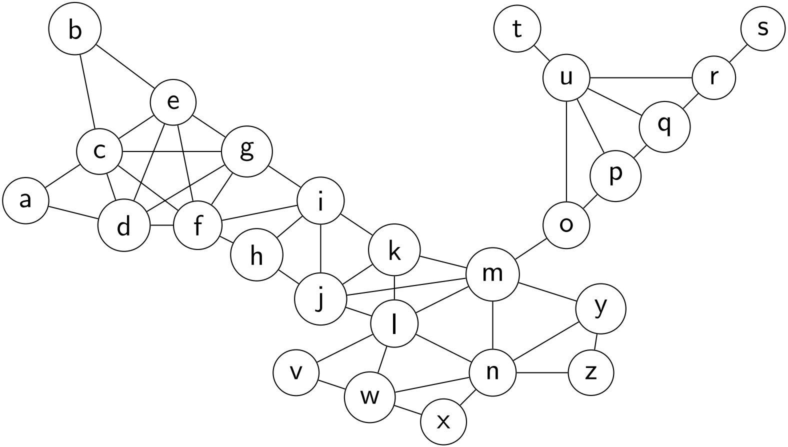
We solve the disjoint compression problem:

CHORDAL EDITING COMPRESSION $(G, M, (k_1, k_2, k_3))$
Input: A graph G, nonnegative integers (k_1, k_2, k_3), a hole cover M of size $\leq k_1 + k_2 + k_3 + 1$.
Task: construct a chordal editing set (V_-, E_-, E_+) such that its size is at most (k_1, k_2, k_3) and V_- is disjoint from M.

$G - M$ is chordal.

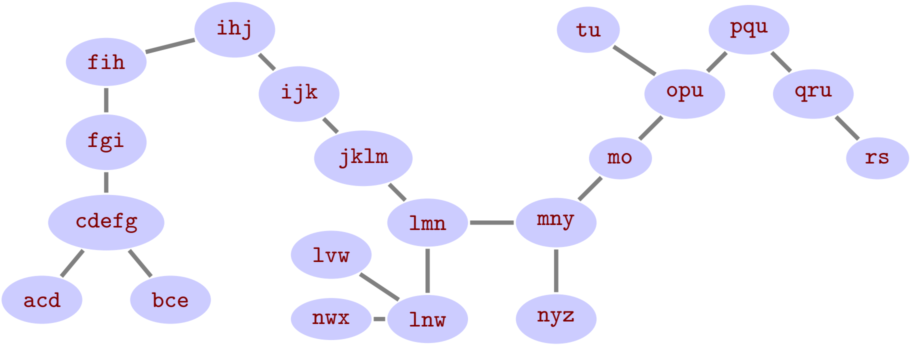
A chordal graph



Theorem (Dirac, 1961)

A chordal graph has at most n maximal cliques.

Clique tree decomposition



Theorem (Dirac, 1961)

1. *Every bag is a maximal clique of G;*
2. *the intersection of two adjacent bags is a minimal separator;*

If it looks like a tree, it probably is a tree.

WEIGHTED FEEDBACK VERTEX SET

Delete vertices of degree 1.

In a chain of degree-2 vertices, only consider the one with min weight.

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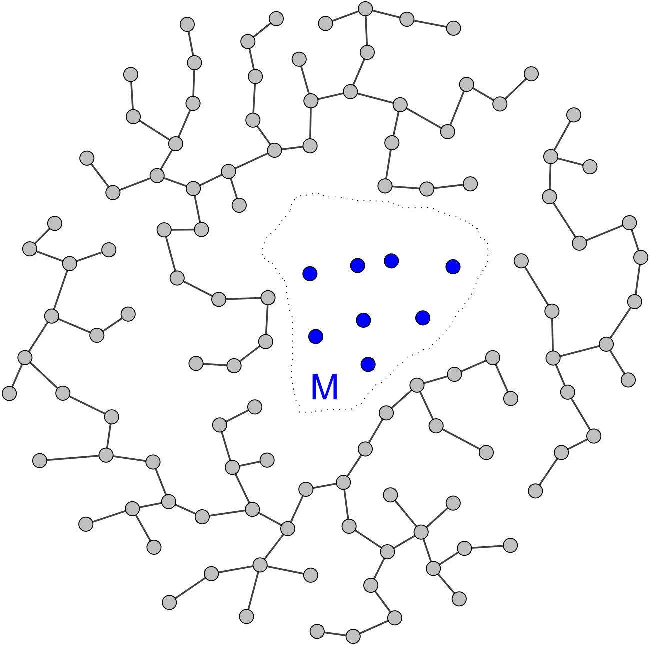
Delete vertices of degree 1.

Delete simplicial vertices ($N[v]$ is a leaf in the clique tree).

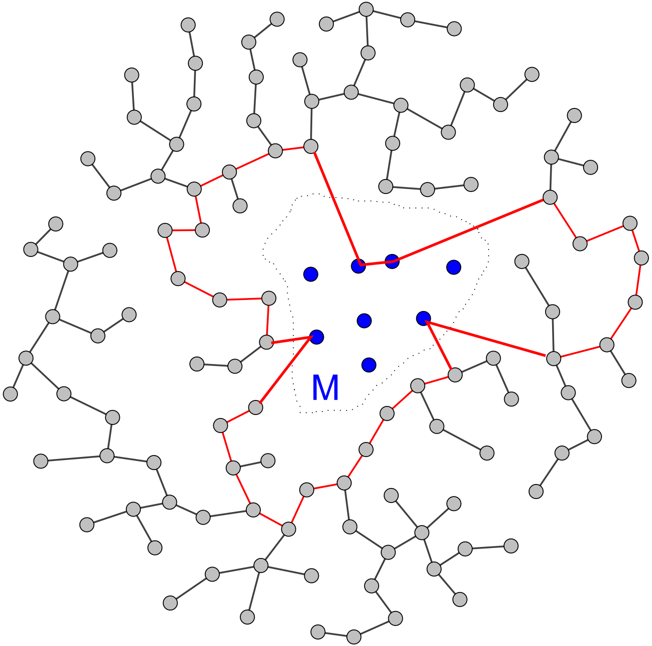
In a chain of degree-2 vertices, only consider the one with min weight.

In a chain of degree-2 bags in the clique tree, only consider the min separator.

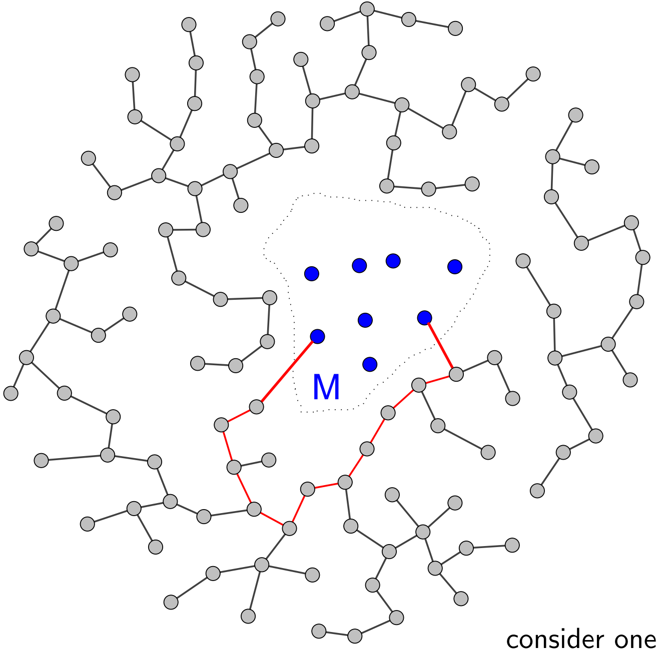
CHORDAL VERTEX DELETION



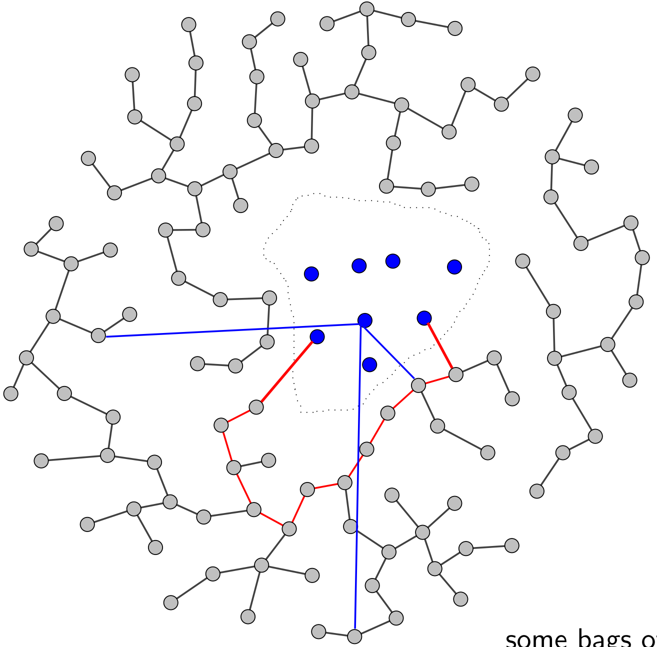
$G - M$ is chordal



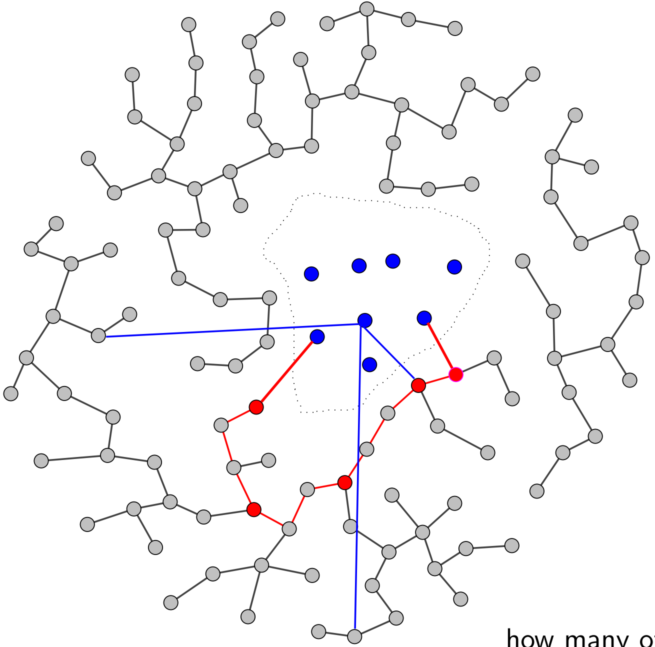
at most $|M|$ paths



consider one particular path P.

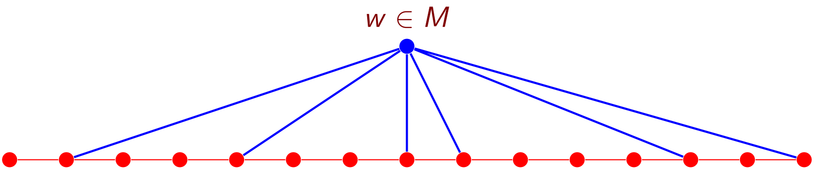


some bags of the **clique** path are **connected** to other vertices in M:



how many of such bags?

Number of segments



- we assume that w is not a common neighbor of H;
- it cannot be adjacent more than 3 vertices in H;
  otherwise we can use w to get a shorter hole than H;
- a pair of neighboring **blue connections** makes a hole;
- these holes share only w;
- we can return "NO" if there are more than $k + 1$ such holes (recall that $V_{-} \cap M = \emptyset$).

Outline of our algorithm

0. **return** if G is chordal or one of k_1, k_2, and k_3 becomes negative;
1. find a shortest hole H;
2. **if** H is shorter than $k + 4$ **then** guess a way to fix it; **goto** 0.
3. **else** decompose H into $O(k^3)$ segments;
   guess a segment and break it;
4. **goto** 0.

Conclusions

1. graph modification problem in the most general sense.
   INTERVAL EDITING? UNIT INTERVAL EDITING?
2. can CHORDAL EDITING be solved in $O(c^k \cdot n^{O(1)})$ time?
   CHORDAL DELETION?
3. does CHORDAL DELETION have a polynomial kernel?

4. what the complexity for finding a shortest hole?