Adaptive Filtering for Stochastic Systems With Generalized Disturbance Inputs

Yan Liang, *Member, IEEE*, Donghua Zhou, *Senior Member, IEEE*, Lei Zhang, *Member, IEEE*, and Quan Pan, *Member, IEEE*

***Abstract*—This letter presents a new class of discrete-time linear stochastic systems with the statistically-constrained disturbance input, which can represent an arbitrary linear combination of dynamic, random, and deterministic disturbance inputs to generalize the complicated modeling error encountered in actual applications. An adaptive filtering scheme is proposed for such systems by recursively constructing and adaptively minimizing the upper-bounds of covariance matrices of the state predictions, innovations, and estimates. The minimum-upper-bound filter is then obtained via online scalar convex optimization. The experiment on maneuvering target tracking shows that the proposed filter can significantly reduce the peak estimation errors due to maneuvers, compared with the well-known IMM method.**

***Index Terms*—Adaptive Kalman filtering, discrete time systems, stochastic systems.**

I. Introduction

As an optimal linear minimum mean square error estimator, the Kalman filter (KF) is widely used in signal processing. However, its performance will degrade greatly if the modeling errors caused by parameter variations and external disturbances cannot be well represented. It has motivated many studies of adaptive filtering in the presence of disturbance inputs (DI) to the system model of KF.

Many strategies were proposed to model DI. First, it is modeled as zero-mean random noise with unknown covariance. The filter design is based on the online covariance identification via Bayesian or maximum likelihood estimation for stationary noise process in linear time-invariant systems [1], [2]. Recently, filter design was extended to time-variant covariance [3] and jump Markov stochastic systems [4]. Second, the DI is assumed to be deterministic. Using least-square estimation and moving-window hypothesis testing, the filters can deal with the DI which is piecewise-constant [5] or a sum of basis functions with piecewise-constant weights [6]. Third, the DI is arbitrary but the rank of its distribution matrix is less than that of measurement matrix. The corresponding filter is an asymptotically-stable and DI-decoupling observer [7]. Fourth, the DI is norm-bounded and the robust filters are designed offline to minimize the gain of the transfer function from DI to the estimation error [8]. Fifth, the DI is a randomly switching parameter obeying a known Markov chain. The filters are multiple model estimators [9]–[13], such as the interacting multiple model (IMM) which is well-known in maneuvering target tracking.

In general, the above filters are DI-specific due to their significant differences in both DIs and solutions. The actual applications, however, encounter much more complicated DIs. It is highly demanded to generalize the DIs and pursue the corresponding filter. To the best of our knowledge, little research has been reported on this topic.

This letter presents a generalized type of DI – a class of statistically constrained DIs, which can represent an arbitrary linear combination of dynamic DIs, random DIs, and deterministic DIs. The proposed filter can adaptively minimize the upper bounds of covariance matrices of the state prediction, filtering residual, and state estimate.

Throughout this letter, for any two square matrices, A and B, $\mathrm{A}-\mathrm{B}\geq 0$ and $\mathrm{A}-\mathrm{B}>0$ mean that A − B is positive semi-definite and positive definite, respectively. Symbol "=:" means definition.

II. Problem Formulation

Consider a new discrete-time linear stochastic system

$$\begin{cases} x_{k+1}=F_kx_k+B_ku_k+\Gamma_kq_k+\delta_k \\ y_{k+1}=H_{k+1}x_{k+1}+v_{k+1} \end{cases} \tag{1}$$

where $x\in R^n$, $u\in R^l$, and $y\in R^m$ represent the system state, control input, and measurement, respectively. The matrices F, B, Γ, and H are known with proper dimension. The process noise $q\in R^p$ and measurement noise $v\in R^m$ are zero-mean white noises with known covariance $Q_k\geq 0$ and $R_{k+1}>0$, respectively. Noises q and v and the initial state x_0 are independent. The introduced new term δ_k satisfies

$$\begin{cases} E\left\{\delta_kq_j^T\right\}=\mathbf{O}_{n\times p} \\ E\left\{\delta_k\nu_{j+1}^T\right\}=\mathbf{O}_{n\times m} \end{cases} \quad (\forall j\geq k) \tag{2}$$

where $O_{n\times p}$ and $O_{n\times m}$ are zero matrices with dimension $n\times p$ and $n\times m$, respectively. Denote by $\overline{Q}^{k-1} =: [q_0^T,\ldots,q_{k-1}^T]^T$, $\overline{V}^k =: [v_1^T,\ldots,v_k^T]^T$, $\overline{\Delta}^{k-1} =: [\delta_0^T\cdots\delta_{k-1}^T]^T$, and $\overline{W}^k =: [\omega_0^T,\ldots,\omega_k^T]^T$, where ω_k is a random noise which is uncorrelated with q and v but whose mean and covariance are unknown. Let f_{1k} be an arbitrary linear time-variant function of $\overline{Q}^{k-1}$, $\overline{V}^k$, and $\overline{\Delta}^{k-1}$; let f_{2k} be an arbitrary deterministic time-variant

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Y. Liang and Q. Pan are with the College of Automation, Northwestern Polytechnical University, Xi'an, China (e-mail: liangyan@nwpu.edu.cn; quanpan@nwpu.edu.cn).

D. H. Zhou is with the Department of Automation, Tsinghua University, Beijing 100081, China (e-mail: zdh@mail.tsinghua.edu.cn).

L. Zhang is with the Department of Computing, The Hong Kong Polytechnic University, Hong Kong (e-mail: cslzhang@comp.polyu.edu.hk).

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function; and let f_k be an arbitrary linear weighted sum of f_{1k}, f_{2k}, and ω_k.

1) Remark 2.1: Repeatedly substituting f_i by f_{i-1} $(i \leq k)$, it is clear that f_k is a linear function of $\overline{Q}^{k-1}$, $\overline{V}^k$, $\overline{W}^k$, and $\overline{\Delta}^{k-1}$. With the fact that $\overline{Q}^{k-1}$, $\overline{V}^k$, $\overline{W}^k$, and $\overline{\Delta}^{k-1}$ are linearly independent of q_j and v_{j+1} $(\forall j \geq k)$, it is concluded that f_k is linearly independent of q_j and v_{j+1}, and thus, $\delta_k = f_k$ satisfies (2). Therefore, δ_k can represent an arbitrary linear weighted sum of f_{1k}, f_{2k}, and ω_k even the corresponding weight matrices are unknown. Here f_{1k}, f_{2k}, and ω_k represent a class of DIs with dynamic property, deterministic DIs, and random DIs, respectively. That is to say, δ_k in (1) is a type of generalized DI (GDI) to represent the complicated modeling error. Because the existing filters are DI-specific and cannot be extended to deal with δ_k, the new filter is required.

III. Upper Bound Filter Design

1) Definition 3.1: A linear filter (3)–(5) for system (1)–(2) is called an upper bound filter (UBF)

$$\text{state prediction } \hat{x}_{k+1|k} = F_k\hat{x}_{k|k} + B_k u_k \tag{3}$$
$$\text{filtering residual } \gamma_{k+1} = y_{k+1} - H_{k+1}\hat{x}_{k+1|k} \tag{4}$$
$$\text{state estimate } \hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\gamma_{k+1} \tag{5}$$

if there exist a sequence of positive-definite matrices $P^*_{k+1|k}$, V^*_{k+1}, and $P^*_{k+1|k+1}$ that satisfy

$$P^*_{k+1|k} \geq P_{k+1|k} =: E\{\widetilde{x}_{k+1|k}\widetilde{x}^T_{k+1|k}\} \tag{6}$$
$$V^*_{k+1} \geq V_{k+1} =: E\left\{\gamma_{k+1}\gamma^T_{k+1}\right\} \tag{7}$$
$$P^*_{k+1|k+1} \geq P_{k+1|k+1} =: E\left\{\widetilde{x}_{k+1|k+1}\ \widetilde{x}^T_{k+1|k+1}\right\} \tag{8}$$

where

$$\text{state prediction error } \widetilde{x}_{k+1|k} =: x_{k+1} - \hat{x}_{k+1|k} \tag{9}$$
$$\text{state estimate error } \widetilde{x}_{k|k} =: x_k - \hat{x}_{k|k} \tag{10}$$

and the filter gain K_{k+1} is a function of $P^*_{k+1|k}$ and V^*_{k+1}.

2) Remark 3.1: Here δ_k cannot be determined from the measurements up to time k and thus cannot be compensated in state prediction. Thus, the expression of the state prediction and measurement prediction (3)–(4) is the same as those of KF with the nominal model [equivalently (1)–(2) with $\delta \equiv 0$]. Because (5) is suitable for any linear estimate, the filter (3)–(5) is general.

3) Remark 3.2: Putting (1), (3), and (10) into (9), the state prediction error is

$$\widetilde{x}_{k+1|k} = F_k\widetilde{x}_{k|k} + \delta_k + \Gamma_k q_k. \tag{11}$$

From (2) and the fact that the zero-mean white noises q and v are independent, it is easy to testify the linear independence between $F_k\widetilde{x}_{k|k} + \delta_k$ and q_k and the linear independence between $\widetilde{x}_{k+1|k}$ and v_{k+1}. Thus, putting (11) into (6) will lead to (12); putting (1), (4), and (9) into (7) will lead to (13); and putting (5) and (8) into (10) will lead to (14) as follows:

$$P_{k+1|k} = E\{[F_k\widetilde{x}_{k|k}+\delta_k][F_k\widetilde{x}_{k|k}+\delta_k]^T\} + \Gamma_k Q_k \Gamma^T_k \tag{12}$$
$$V_{k+1} = H_{k+1}P_{k+1|k}H^T_{k+1} + R_{k+1} \tag{13}$$
$$\begin{aligned} P_{k+1|k+1} = &(I - K_{k+1}H_{k+1})P_{k+1|k}(I - K_{k+1}H_{k+1})^T \\ &+ K_{k+1}R_{k+1}K^T_{k+1}. \end{aligned} \tag{14}$$

The GDI δ_k in (12) is the barrier to filter implementation. It is not feasible to directly estimate $P_{k+1|k}$ with $n \times (n+1)/2$ independent parameters except for $n = 1$ because the measurement at time k can only supply at most n independent equations for possible parameter estimation. This is based on the fact that the rank of the $m \times n$ measurement matrix is less than $n + 1$. Therefore, it is infeasible to design the filter by estimating the covariance of system (1)–(2). The idea of the UBF design is motivated by the fact that determining the upper bound, instead of the covariance, requires fewer parameters to be estimated.

As shown in (12), δ_k appears in the covariance of the state prediction in UBFs, as the system uncertainty. Thus, we have

$$\begin{aligned} P_{k+1|k} &\geq E\left\{F_k\widetilde{x}_{k|k}\widetilde{x}^T_{k|k}F^T_k\right\} + \Gamma_k Q_k \Gamma^T_k \\ &= F_k P_{k|k} F^T_k + \Gamma_k Q_k \Gamma^T_k. \end{aligned}$$

Meanwhile, V_{k+1} and $P_{k+1|k+1}$ have nothing to do with δ_k given $P_{k+1|k}$. Hence, the following recursive upper-bound structure is considered:

$$P^*_{k+1|k} = \alpha_k F_k P^*_{k|k} F^T_k + \Gamma_k Q_k \Gamma^T_k \tag{15}$$
$$V^*_{k+1} = H_{k+1}P^*_{k+1|k}H^T_{k+1} + R_{k+1} \tag{16}$$
$$\begin{aligned} P^*_{k+1|k+1} = &(I - K_{k+1}H_{k+1})P^*_{k+1|k}(I - K_{k+1}H_{k+1})^T \\ &+ K_{k+1}R_{k+1}K^T_{k+1} \end{aligned} \tag{17}$$

where the fading factor $\alpha_k \geq 1$ is a parameter to be estimated.

It is needed to know whether there exists a UBF based on (15)–(17). If a UBF exists, it is further needed to determine the optimal filter parameters through minimizing the upper bounds so that the minimum UBF (MUBF) can be obtained. The following theorem provides the solution.

4) Theorem 3.1: If the following three conditions are satisfied:

$$\text{(i) } P^*_{0|0} \geq P_{0|0} \tag{18}$$
$$\text{(ii) } H_k \text{ is of full column rank, i.e.,}$$
$$rank\{H_k\} = n,\ \forall k \tag{19}$$
$$\text{(iii) } V^*_{k+1} \geq V_{k+1} \tag{20}$$

then there exists a UBF with structure (15)–(17) and optimal parameters α^{Opt}_k and K^{Opt}_{k+1}. For any α_k satisfying $V_{k+1} \leq V^*_{k+1}\big|_{\alpha_k}$ and any filter gain $K(k+1)$, there exist

$$P_{k+1|k} \leq P^*_{k+1|k}\Big|_{\alpha^{Opt}_k} \leq P^*_{k+1|k}\Big|_{\alpha_k} \tag{21}$$
$$V_{k+1} \leq V^*_{k+1}\big|_{\alpha^{Opt}_k} \leq V^*_{k+1}\big|_{\alpha_k} \tag{22}$$
$$\begin{aligned} P_{k+1|k+1}\big|_{K_{k+1}} &\leq P^*_{k+1|k+1}\Big|_{\alpha^{Opt}_k, K^{Opt}_{k+1}} \\ &\leq P^*_{k+1|k+1}\Big|_{\alpha_k, K_{k+1}} \end{aligned} \tag{23}$$

and the optimal filter parameters are

$$\alpha^{Opt}_k = \min\{\alpha_k | \alpha_k \in \Lambda_k\} \tag{24}$$
$$K^{Opt}_{k+1} = P^*_{k+1|k}H^T_{k+1}V^{*-1}_{k+1} \tag{25}$$

where $\Lambda_k =: \left\{\alpha_k | \alpha_k \geq 1,\ V^*_{k+1}\big|_{\alpha_k} \geq V_{k+1}\right\}$. ■

Proof: See the Appendix . ■

5) Remark 3.3: In the traditional KF, $P_{0|0}$ is known *a priori*. In the UBF proposed in this letter, only its upper bound $P_{0|0}^*$ is needed. Constraint (19) requires that the current measurement of system (1)–(2) should provide enough information for disturbance compensation. The third condition of Theorem 3.1 means that the fading factor α_k should be large enough to guarantee the existence of (7).

Putting (13) and (16) into (24), the optimal fading factor is the solution to the following scalar convex optimization:

$$\alpha_k^{Opt} = \min \alpha_k \tag{26}$$

which is subject to
1) $\alpha_k \geq 1$
2) $\alpha_k H_{k+1} F_k P_{k|k}^* F_k^T H_{k+1}^T + H_{k+1} \Gamma_k Q_k \Gamma_k^T H_{k+1}^T + R_{k+1} \geq V_{k+1}$.

In (26), V_{k+1} is unknown and thus substituted by its unbiased estimate $\hat{V}_{k+1} =: \gamma_{k+1}\gamma_{k+1}^T$. Then linear matrix inequality (LMI) [14] is used to solve (26). The MUBF is expected robust to such approximation due to its robustness to parameter inaccuracy in pursuing the best solution in the "worst" possible case.

The recursive MUBF algorithm is summarized as follows: compute $\hat{x}_{k+1|k}$ by (3) and γ_{k+1} by (4); then determine α_k^{Opt} via LMI optimization using (26), where V_{k+1} is substituted by $\hat{V}_{k+1}$; compute $P_{k+1|k}^*\big|_{\alpha_k^{Opt}}$ by (15), $V_{k+1}^*\big|_{\alpha_k^{Opt}}$ by (16), K_{k+1}^{Opt} by (25), $\hat{x}_{k+1|k+1}$ by (5), and $P_{k+1|k+1}^*\big|_{\alpha_k^{Opt}, K_{k+1}^{Opt}}$ by (17).

IV. Experiment

The experiment of maneuvering target tracking in the benchmark target tracking scenario [10] was performed to evaluate the proposed MUBF in comparison with the well-known IMM [9]. The position, velocity, and acceleration of a maneuvering target in a two-dimensional ξ-η plane are shown in Fig. 1. The sampling period is 1 s. The observation is

$$z_{k+1} = \begin{bmatrix} \sqrt{\xi_{k+1}^2 + \eta_{k+1}^2} \\ \arctan\left(\frac{\eta_{k+1}}{\xi_{k+1}}\right) \end{bmatrix} + \nu_{k+1} \tag{27}$$

where ν_{k+1} is zero-mean Gaussian measurement noise with covariance $R_{k+1} = diag\{{\sigma_1}^2, {\sigma_2}^2\}$. Through the first-order linearization of (27), as in the extended Kalman filter (EKF), the parameters of measurement equation in (1) are

$$y_{k+1} = z_{k+1} - \begin{bmatrix} \sqrt{\hat{\xi}_{k+1|k}^2 + \hat{\eta}_{k+1|k}^2} \\ \arctan\left(\frac{\hat{\eta}_{k+1|k}}{\hat{\xi}_{k+1|k}}\right) \end{bmatrix}$$
$$H_{k+1} = \left.\begin{bmatrix} \frac{\partial\left(\sqrt{\xi_{k+1}^2+\eta_{k+1}^2}\right)}{\partial x^T} \\ \frac{\partial\left(\arctan\left(\frac{\eta_{k+1}}{\xi_{k+1}}\right)\right)}{\partial x^T} \end{bmatrix}\right|_{x=\hat{x}_{k+1|k}}.$$

Here the design of IMM is the same as that in [10]. The proposed MUBF treats the unknown maneuver changes in velocity as GDIs to a nominal constant-velocity model. In MUBF design, the state equation in (1) has $x = [\xi\ \dot{\xi}\ \eta\ \dot{\eta}]^T$, $F = diag\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}$. $B_k u_k$ and $\Gamma_k q_k$ are 4×1 zero vectors and $\delta_k = x_{k+1} - F_k x_k$. It should be stressed that the process noise is set to be zero in the MUBF. That is to say, unlike in IMM, the MUBF supposes that the target maneuver is unknown *a priori* and treats it completely as GDI. In maneuvering target tracking, the tracking loss comes mainly from the abrupt maneuvering actions, which lead to significant uncertainties in target position. Therefore, the peak root-mean-square-error (RMSE) in the position prediction and estimation are the most important indexes for filter performance evaluation. Fig. 2 shows the RMSE of the IMM and MUBF via 1000 Monte Carlo simulations. Compared with the IMM, the MUBF reduces greatly the peak RMSE in target position estimation and prediction due to abrupt maneuvers.

The simulation PC is a HP nc4010. The programming language is Matlab 6.5 with LMI solver "mincx.m". The average running time for each simulation is 6.06 ms in MUBF and 6.70 ms in IMM. The 10% computation burden is saved.

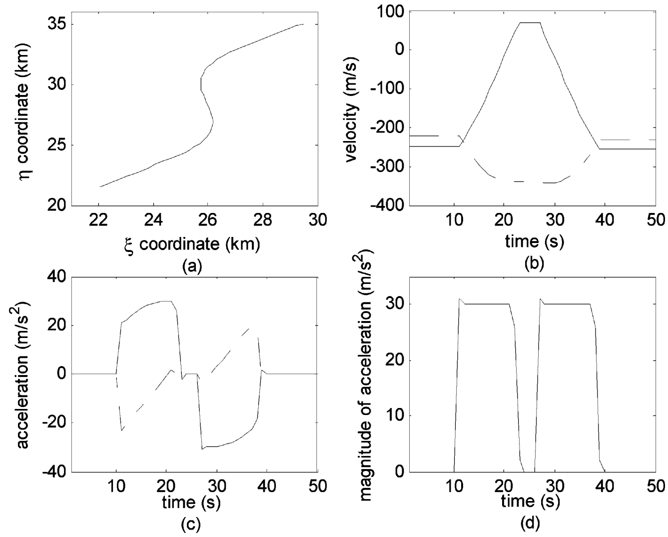

Fig. 1. State to be estimated in the orthogonal ξ-η (Cartesian) coordinates. (a) Position trajectory in ξ-η coordinates. (b) and (c) Velocity and acceleration trajectories in ξ (solid line) and η (dashed line). (d) Acceleration magnitude.

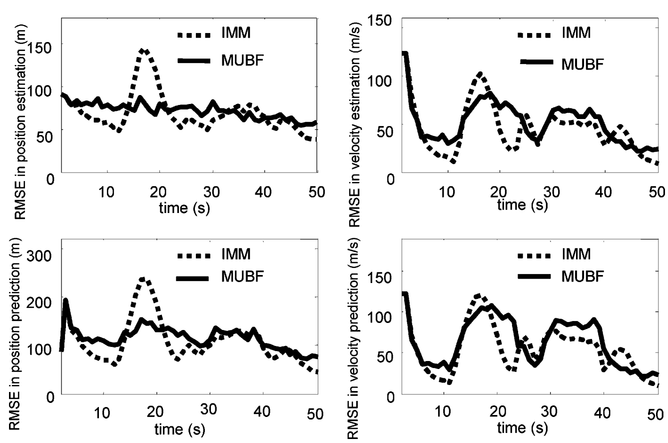

Fig. 2. RMSE of position estimation and prediction.

V. Conclusion

This letter proposed a new discrete-time linear stochastic system model with the statistically-constrained DI, to which the existing DI-specific filters cannot be applied. Through constructing and minimizing the upper bounds of covariance matrices of state prediction, filtering residual, and state estimate, the MUBF was presented with adaptive filter parameters. The simulation of maneuvering target tracking showed that MUBF can significantly decrease the peak error in position, compared with the well-known IMM method.

APPENDIX
PROOF OF THEOREM 3.1

Existence of UBF: It is needed to testify the existence of (6) and (8) because the third condition (7) of Theorem 3.1 is loose. As the first condition of Theorem 3.1, $P_{0|0}^{*} \geq P_{0|0}$ is guaranteed. Using mathematical induction, we assume $P_{k|k}^{*} \geq P_{k|k}$ and testify (6) and (8). Substituting (13) and (16) into (20), there is

$$H_{k+1}[P_{k+1|k}^{*} - P_{k+1|k}]\, H_{k+1}^{T} \geq 0. \tag{A1}$$

Left multiply H_{k+1}^{T} and right multiply H_{k+1} in both sides of (A1), and there is

$$H_{k+1}^{T} H_{k+1}[P_{k+1|k}^{*} - P_{k+1|k}]\, H_{k+1}^{T} H_{k+1} \geq 0. \tag{A2}$$

Because the rank of $H_{k+1}^{T} H_{k+1}$ equals to that of H_{k+1}, and the rank is n according to the second condition of Theorem 3.1, it is concluded that $H_{k+1}^{T} H_{k+1}$ is of full rank. Left and right multiply in both sides of (A.2), and we obtain (6). Substituting (6) into (17) leads to

$$P_{k+1|k+1}^{*} \geq (I - K_{k+1}H_{k+1})P_{k+1|k}(I - K_{k+1}H_{k+1})^{T} + K_{k+1}R_{k+1}K_{k+1}^{T} \tag{A3}$$

and (8) is obtained from (A3) and (14).

Optimal Filter Parameters: According to Definition 3.1, the set $\{\alpha_k \mid V_{k+1}^{*} \geq V_{k+1}\}$ will not be empty if a UBF exists. For any $\alpha_{1k} \in \Lambda_k$, if $\alpha_{1k} \leq \alpha_{2k}$, then $\alpha_{2k} \in \Lambda_k$ because $V_{k+1}^{*}\big|_{\alpha_{1k}} \leq V_{k+1}^{*}\big|_{\alpha_{2k}}$. Thus

$$\begin{aligned}\Lambda_k &= \{\alpha_k \mid \alpha_k \geq 1, V_{k+1}^{*} \geq V_{k+1}\} \\ &= \{\alpha_k \mid \alpha_k \geq 1\} \cap \{\alpha_k V_{k+1}^{*} \geq V_{k+1}\}\end{aligned}$$

is not empty, and hence, there exists $\alpha_k^{Opt} =: \min\{\alpha_k \mid \alpha_k \in \Lambda_k\}$. It is only needed to testify that α_k^{Opt} and K_{k+1}^{Opt} guarantee (21)–(23). From the definition of α_k^{Opt}, $\alpha_k \geq \alpha_k^{Opt}$ exists to any $\alpha_k \in \Lambda_k$. Thus

$$\begin{aligned}\Delta P_k =:\, & P_{k+1|k}^{*}\Big|_{\alpha_k} - P_{k+1|k}^{*}\Big|_{\alpha_k^{Opt}} \\ =\, & (\alpha_k - \alpha_k^{Opt})F_k P_{k|k}^{*} F_k^{T} \geq 0 \end{aligned} \tag{A4}$$

$$\begin{aligned} & V_{k+1}^{*}\big|_{\alpha_k} - V_{k+1}^{*}\big|_{\alpha_k^{Opt}} \\ & = (\alpha_k - \alpha_k^{Opt})H_{k+1}F_k P_{k|k}^{*} F_k^{T} H_{k+1}^{T} \geq 0 \end{aligned} \tag{A5}$$

$$\begin{aligned} & P_{k+1|k+1}^{*}\Big|_{\alpha_k, K_{k+1}} - P_{k+1|k+1}^{*}\Big|_{\alpha_k^{Opt}, K_{k+1}} \\ & = (I - K_{k+1}H_{k+1})\Delta P_k (I - K_{k+1}H_{k+1})^{T} \\ & \geq 0. \end{aligned} \tag{A6}$$

In deriving (A4) and (A6), the expressions of $P_{k+1|k}^{*}$ in (15) and $P_{k+1|k+1}^{*}$ in (17) are used, respectively. To obtain (A5), both expressions of $P_{k+1|k}^{*}$ in (15) and V_{k+1}^{*} in (16) are used.

From (16) and $R_{k+1} > 0$, $V_{k+1}^{*} \geq R_{k+1} > 0$ exists. Represent the symmetric and positive definite matrix $V_{k+1}^{*}\big|_{\alpha_k^{Opt}}$ by $S_{k+1}S_{k+1}^{T}$, where S_{k+1} is of full rank. Letting $D_{k+1} =: P_{k+1|k}^{*}\Big|_{\alpha_k^{Opt}} H_{k+1}^{T} S_{k+1}^{-T}$, there is

$$\begin{aligned} & P_{k+1|k+1}^{*}\Big|_{\alpha_k^{Opt}, K_{k+1}} \\ & = P_{k+1|k}^{*}\Big|_{\alpha_k^{Opt}} - D_{k+1}D_{k+1}^{T} \\ & \quad + (K_{k+1}S_{k+1} - D_{k+1})(K_{k+1}S_{k+1} - D_{k+1})^{T} \\ & \geq P_{k+1|k}^{*}\Big|_{\alpha_k^{Opt}} - D_{k+1}D_{k+1}^{T} = P_{k+1|k+1}^{*}\Big|_{\alpha_k^{Opt}, K_{k+1}^{Opt}} \end{aligned} \tag{A7}$$

where

$$K_{k+1}^{Opt} = D_{k+1}S_{k+1}^{-1} = P_{k+1|k}^{*} H_{k+1}^{T} V_{k+1}^{*-1}. \tag{A8}$$

With the fact that $(K_{k+1}S_{k+1} - D_{k+1})(K_{k+1}S_{k+1} - D_{k+1})^{T} \geq 0$, the inequality (A7) is obtained. Thus, $P_{k+1|k+1}^{*}\Big|_{\alpha_k^{Opt}, K_{k+1}}$ reaches its minimum (optimal) value if and only if (A8) is satisfied. As shown above, a UBF exists and $\alpha_k^{Opt} \in \Lambda_k$. Thus, (6)–(8) are obtained and (21)–(23) are further obtained based on (6)–(8) and (A4)–(A7).

REFERENCES

[1] R. K. Mehra, "Approaches to adaptive filtering," *IEEE Trans. Autom. Control*, vol. AC-17, no. 5, pp. 693–698, Dec. 1972.

[2] K. A. Myers and B. D. Tapley, "Adaptive sequential estimation with unknown noise statistics," *IEEE Trans. Autom. Control*, vol. AC-21, no. 4, pp. 520–523, Aug. 1976.

[3] Y. Liang, D. X. An, D. H. Zhou, and Q. Pan, "A finite horizon adaptive Kalman filter for linear systems with unknown disturbances," *Signal Process., Elsevier*, vol. 84, no. 11, pp. 2175–2194, 2004.

[4] Y. Liang, D. X. An, D. H. Zhou, and Q. Pan, "Estimation of time-varying time delay and parameters of a class of jump Markov nonlinear stochastic systems," *Comput. Chem. Eng., Elsevier*, vol. 27, no. 12, pp. 1761–1778, 2003.

[5] P. L. Bogler, "Tracking a maneuvering target using input estimation," *IEEE Trans. Aerosp. Elect. Syst.*, vol. 23, no. 3, pp. 298–310, Jul. 1987.

[6] H. Lee and M. J. Tahk, "Generalized input-estimation technique for tracking maneuvering targets," *IEEE Trans. Aerosp. Elect. Syst.*, vol. 35, no. 4, pp. 1388–1402, Oct. 1999.

[7] M. Darouach and M. Zasadzinski, "Unbiased minimum variance estimation for systems with unknown exogenous inputs," *Automatica*, vol. 33, no. 4, pp. 717–719, Apr. 1997.

[8] L. E. Ghaoui and G. Calafiore, "Robust filtering for discrete-time systems with bounded noise and parametric uncertainty," *IEEE Trans. Autom. Control*, vol. 46, no. 7, pp. 1084–1089, Jul. 2001.

[9] E. Mazor, A. Averbuch, Y. Bar-Shalom, and J. Dayan, "Interacting multiple model methods in multiple-multiple tracking: A survey," *IEEE Trans. Aerosp. Elect. Syst.*, vol. 34, no. 1, pp. 103–122, Feb. 1998.

[10] B. Chen, "Interacting multiple model fixed-lag smoothing algorithm for Markovian switching systems," *IEEE Trans. Aerosp. Elect. Syst.*, vol. 36, no. 1, pp. 243–250, Feb. 2000.

[11] Z. Wang, J. Lam, and X. Liu, "Robust filtering for discrete-time Markovian jump delay systems," *IEEE Signal Process. Lett.*, vol. 11, no. 8, pp. 659–662, Aug. 2004.

[12] L. A. Johnston and V. Krishnamurthy, "An improvement to the interacting multiple model (IMM) algorithm," *IEEE Trans. Signal Process.*, vol. 49, no. 12, pp. 2909–2923, Dec. 2001.

[13] S. K. Katsikas, S. D. Likothanassis, G. N. Beligiannis, K. G. Berkeris, and D. A. Fotakis, "Genetically determined variable structure multiple model estimation," *IEEE Trans. Signal Process.*, vol. 49, no. 10, pp. 2253–2261, Oct. 2001.

[14] G. Tadmor and L. Mirkin, "H∞ control and estimation with preview–Part II: Fixed-size are solutions in discrete time," *IEEE Trans. Autom. Control*, vol. 50, no. 1, pp. 29–40, Jan. 2005.