An $O(k^4)$ Kernel for Unit Interval Vertex Deletion

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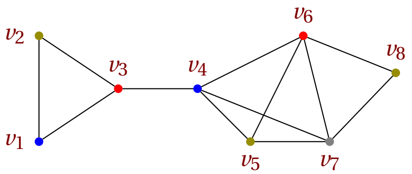
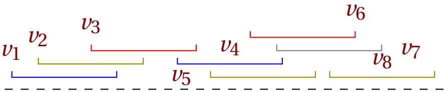
Unit interval graphs

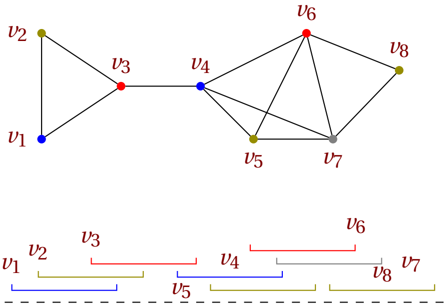


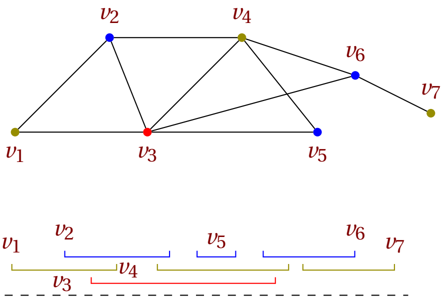


Definition

There are a set of unit-length intervals $\mathscr{I}$ on the real line and $\phi : V \to \mathscr{I}$ such that $uv \in E(G)$ iff $\phi(u)$ intersects $\phi(v)$.

Unit interval graphs





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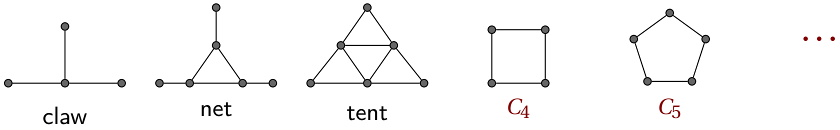
Forbidden induced subgraphs

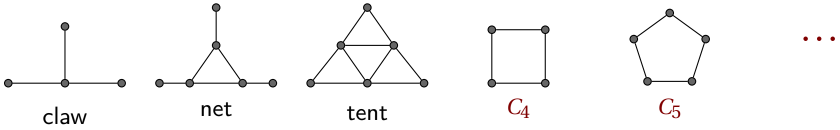
[Wegner 1967]



unit interval ⊂ interval ⊂ chordal (hole-free)

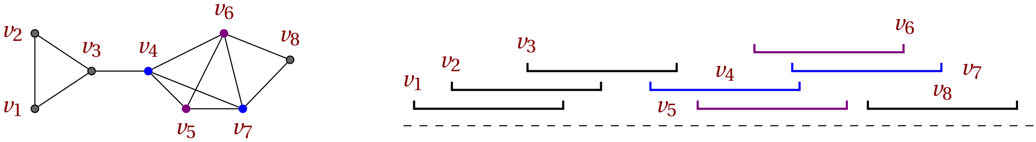
Forbidden induced subgraphs

[Wegner 1967]



unit interval $\subset$ interval $\subset$ chordal (hole-free)

Proper interval ordering



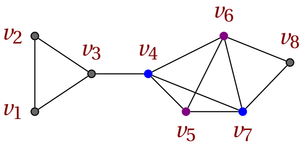
Theorem (Looges 1993)

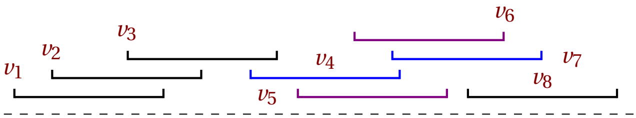
G is a unit interval graph iff there exists an ordering $\{v_1, \ldots, v_n\}$ such that for every $1 \le i < j < k \le n$, $v_i v_k \in E(G)$ implies $v_i v_j$, $v_j v_k \in E(G)$.

The ordering of the left (right) endpoints of the intervals will do.

Corollary

If $1 \le i < j \le n$ and $v_i v_j \in E(G)$, then $\{v_i, \cdots, v_j\}$ is a clique.





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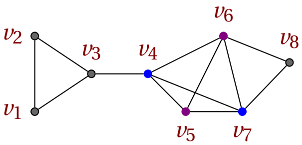

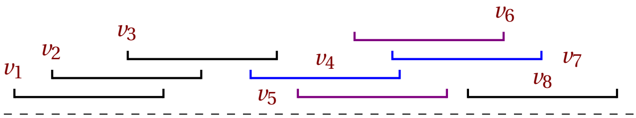

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The problem

Unit interval vertex deletion

Input: A graph G and an integer k.

Task: A set V_- of $\leq k$ vertices such that $G - V_-$ is a unit interval graph.

$$\mathrm{NP-complete}$$

[Lewis & Yannakakis 1980]

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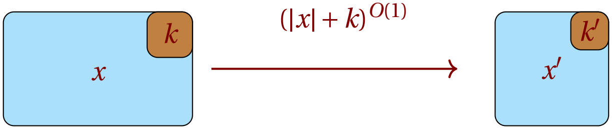
[Lewis & Yannakakis 1980]

FPT	$O((14k+14)^{k+1} \cdot kn^6)$	$O(6^k \cdot n^6)$	$O(6^k \cdot (n+m))$
[Marx 2006]	[van Bevern et al. 2010]	[Villanger 2010]	[C 2015]

Kernelization

Definition

Given an instance (x, k), a *kernelization algorithm* produces an equivalent instance (x', k') with $|x'| = O(f(k))$ and $k' \leq k$ in time $(|x| + k)^{O(1)}$.



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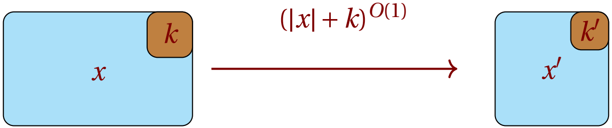

$O(k^{53})$
[Fomin, Saurabh, Villanger 2013]

$O(k^4)$
[this talk]

Main ideas

- Use the 6-approximation algorithm to produce a modulator M.
- Partition the vertices in $G - M$, a unit interval graph, into $O(k^2)$ cliques.
- Pick $O(k^3)$ vertices from each clique to make the kernel.
- $O(k^5) \to O(k^4)$, with more refined counting.

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The modulator

Theorem (C 2015)

There is a 6-approximation for unit interval vertex deletion.

- We start by founding an approximate solution M to G.
- If $|M| > 6k$, then return a trivial no-instance.
- Henceforth, $G - M$ is a unit interval graph, where $|M| \leq 6k$.

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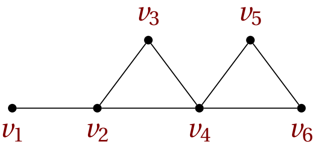
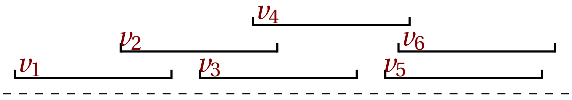
The clique partition

- Find a unit interval model for $G - M$. [Corneil 2004]
- Choose the first unassigned vertex, and all its unassigned neighbors; repeat till all vertices assigned.





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2. for $i < j$, the distance between $u \in K_i$ and $v \in K_j$ is $\geq j - i$.

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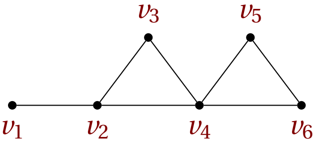

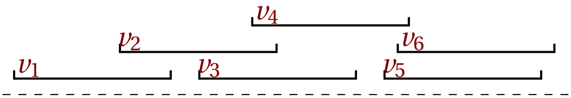

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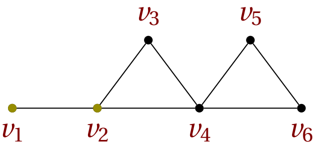

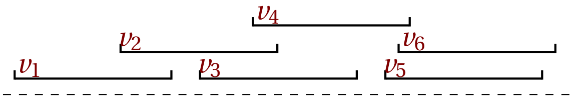

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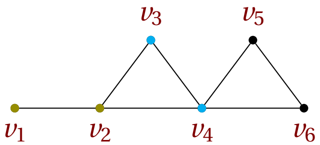

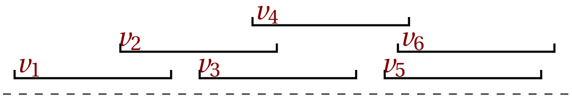

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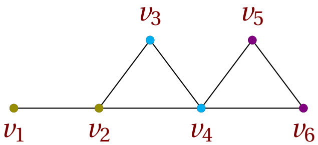

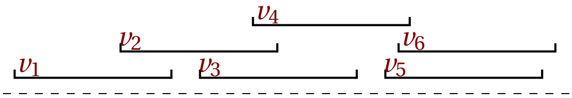

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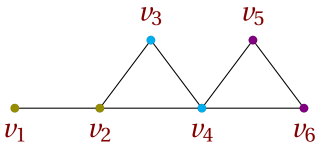

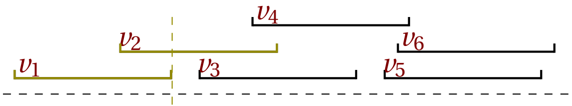

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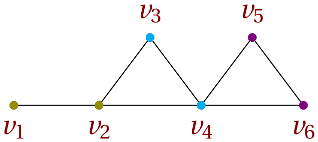

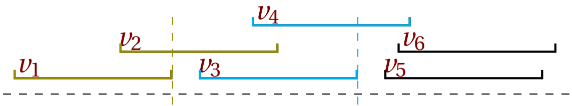

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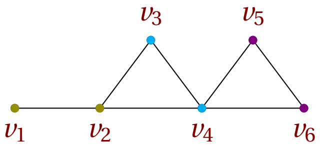

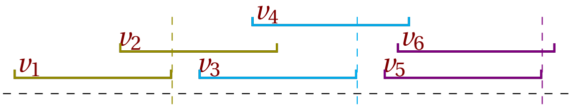

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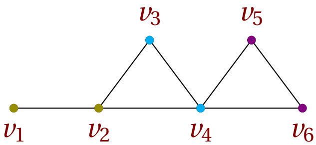

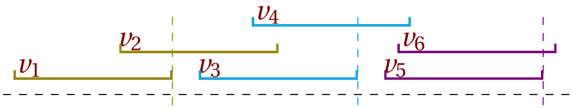

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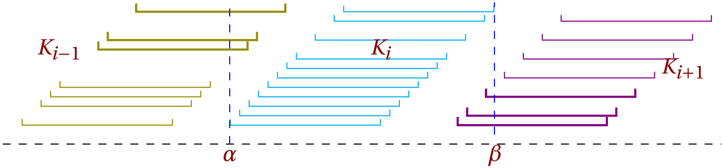
The Number of Cliques

Bypassing a clique



delete K_i, and
add all possible edges between
$N(K_i) \cap K_{i-1}$ and $N(K_i) \cap K_{i+1}$.

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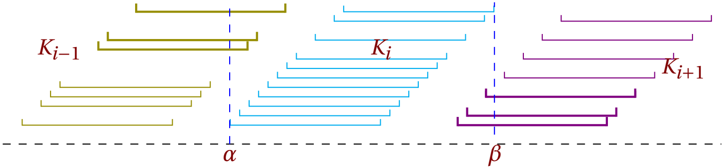

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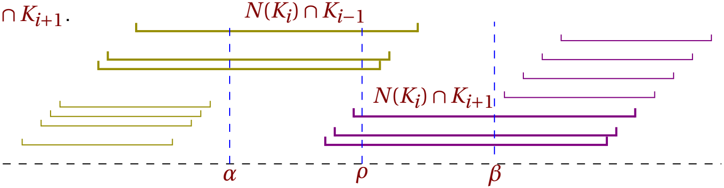

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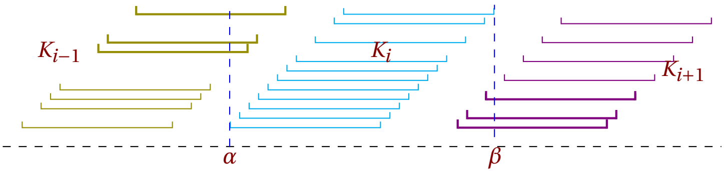

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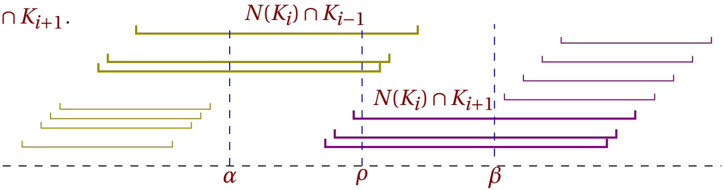


Lemma

The graph obtained by bypassing a clique in the partition is still a unit interval graph.

Cliques adjacent to M

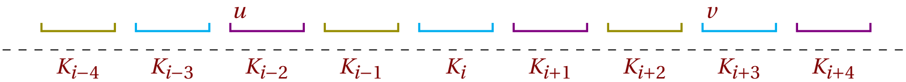
Rule 1. If a vertex $v \in M$ has neighbors in $\geq k+5$ cliques, then $(G,k) \to (G-\{v\}, k-1)$

Proof.

There is a claw if v has neighbors in ≥ 5 cliques. □

Cliques not adjacent to M

9 consecutive cliques nonadjacent to M



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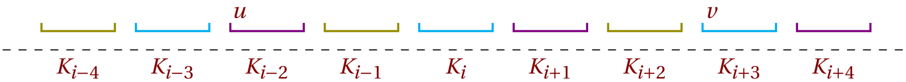

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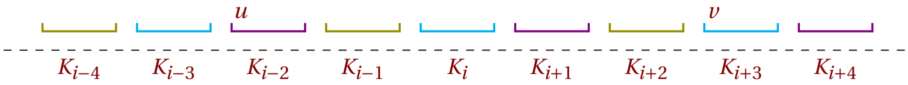

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Because its distance to M is at least 4.

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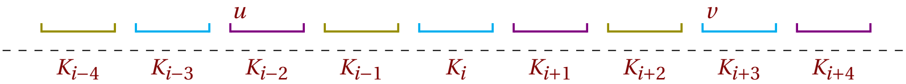

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Because its distance to M is at least 4.

Rule 2. Find a minimum $u-v$ separator S in $G-M$.
One of K_{i-1} and K_{i+1} is disjoint from S, which is a clique; bypass it.

The number of cliques

Rule 1 bounds the number of cliques containing neighbors of M.
$|M|(k+4) \leq 6k^2 + 24k$.

Rule 2 bounds the number of cliques lying between them.

Lemma.
If neither of Rule 1, 2 is applicable, then the number of cliques (in $G - M$) is $O(k^2)$.

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Irrelevant Vertices

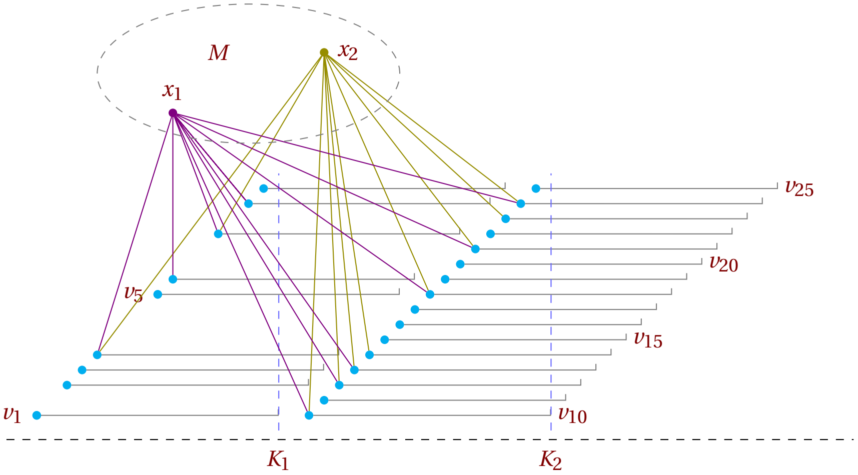
Pick a vertex set $U \subseteq V(G) \setminus M$ such that

any solution X for $G[U \cup M]$ is also a solution for G.

The set U

- for each pair $x_1, x_2 \in M$ and each $i = 1, \ldots, t$, we pick the first/last $k+1$ vertices from K_i for each of the four patterns—adjacent to both; adjacent to only x_1; adjacent to only x_2; and adjacent to neither.
- for each $x \in M$, each $i = 2, \ldots, t$, and each y of the last $k+1$ non-neighbors of x in K_{i-1}, we pick the last $k+1$ common neighbors of x and y in K_i.
- for each $x \in M$, each $i = 2, \ldots, t$, and each y of the first $k+1$ neighbors of x in K_{i-1}, we pick the first $k+1$ vertices in K_i that are neighbors of x but not y
- for each $x \in M$, each $i = 2, \ldots, t$, and each y of the last $k+1$ neighbors of x in K_{i-1}, we pick the last $k+1$ vertices in K_i that are neighbors of y but not x.
- for each three pairwise nonadjacent vertices in M, we arbitrarily pick $k+1$ common neighbors of them in $V(G) \setminus M$.
- For each triple of vertices in M that induces a P_3, we arbitrarily pick $k+1$ vertices in $V(G) \setminus M$ that are adjacent to only the center vertex among them, and $k+1$ vertices in $V(G) \setminus M$ that are nonadjacent to only the center vertex among them.

For each pair $x_1, x_2 \in M$, pick the first/last $k+1$ vertices from K_i that are adjacent to (1) both x_1 and x_2; (2) only x_1; (3) only x_2; and (4) neither of them.



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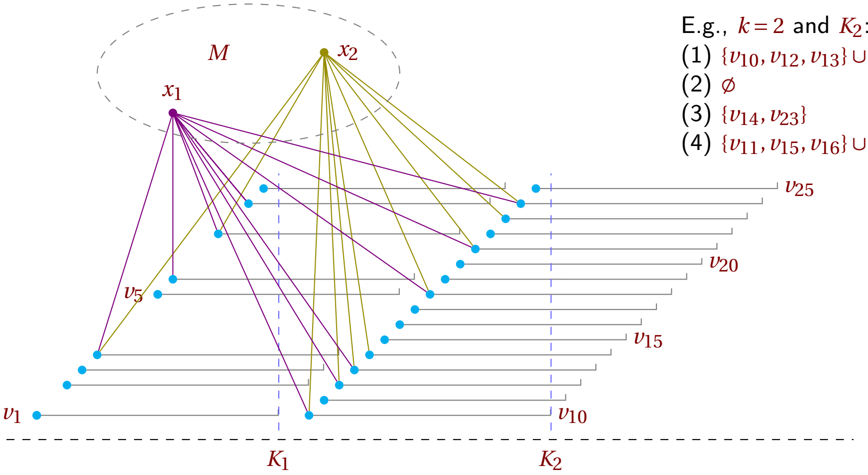


E.g., $k=2$ and K_2:

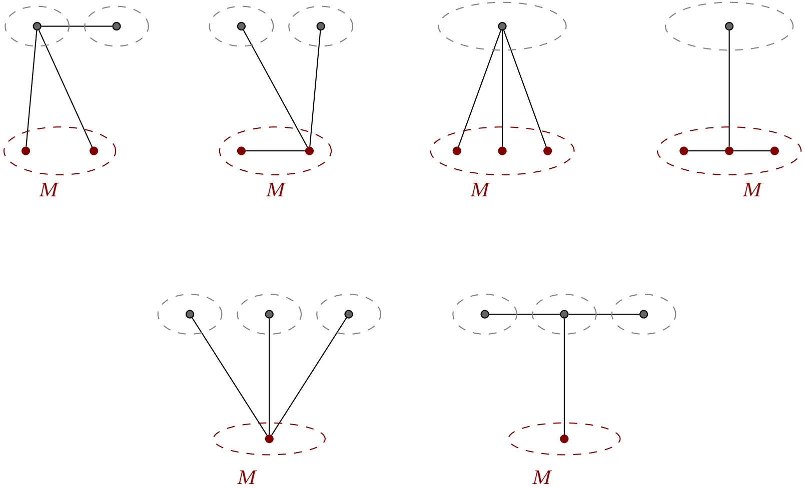
(1) $\{v_{10}, v_{12}, v_{13}\} \cup \{v_{18}, v_{21}, v_{24}\}$

(2) $\emptyset$

(3) $\{v_{14}, v_{23}\}$

(4) $\{v_{11}, v_{15}, v_{16}\} \cup \{v_{20}, v_{22}, v_{25}\}$.

Claw



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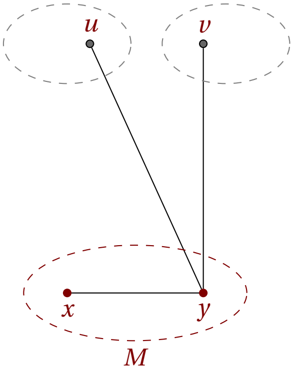

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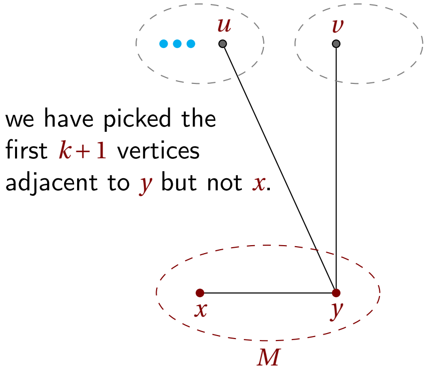

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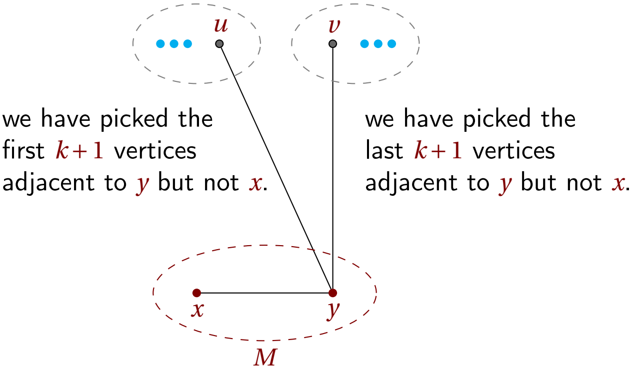

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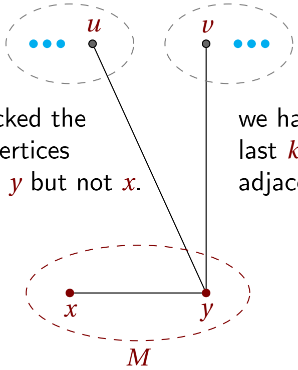

we have picked the first $k+1$ vertices adjacent to y but not x.

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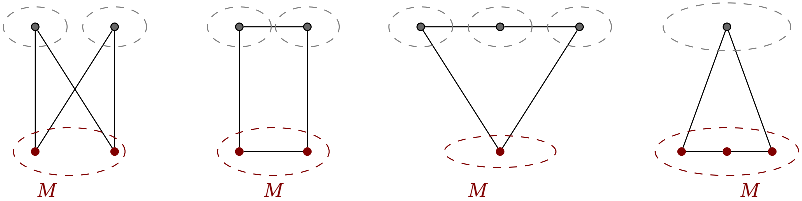
if u is not picked, then at least $k+1$ claws containing x,y,v, of which one has to be in a solution of $G[U \cup M]$.

Similar arguments work for other configurations and for C_4's.



Other obstructions

We may use similar arguments for nets, tents, and longer holes.

But, such an exhaustive case analysis would be long and excruciatingly hard to verify.
(For example, a long hole may go through M many times.)

Instead, we use a constructive argument for them:
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Final remark:
It can be produced in $O(nm)$ time.

Thanks!

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